

The development of elliptic functions according to Ramanujan and Venkatachaliengar

Shaun Cooper

I.I.M.S., Massey University Albany Campus, Auckland, N.Z.

s.cooper@massey.ac.nz

Abstract

These notes are based on the monograph *Development of Elliptic Functions according to Ramanujan* by K. Venkatachaliengar [2]. The goal of the notes is to show how some of the main properties of Jacobian and Weierstrass elliptic functions can be developed in an elementary way from the ${}_1\psi_1$ function.

All of the ideas presented in these notes can be found in Venkatachaliengar's book. The only thing I have done is to rearrange the order in which the material is presented. I am entirely responsible for any errors in these notes, and would be very grateful to be informed about them, whether they be large or small.

1 Introduction

Throughout these notes, let τ be a fixed complex number which satisfies $\text{Im } \tau > 0$ and let $q = e^{i\pi\tau}$, so that $|q| < 1$. We will make use of the following notation for products. Let

$$\begin{aligned}(a; q)_n &= \prod_{j=0}^{n-1} (1 - aq^j), \\ (a; q)_\infty &= \prod_{j=0}^{\infty} (1 - aq^j), \\ (a_1, a_2, \dots, a_n; q)_\infty &= (a_1; q)_\infty (a_2; q)_\infty \dots (a_n; q)_\infty.\end{aligned}$$

We will use the symbol \sum_n to denote summation over all integer values of n from $-\infty$ to ∞ , and \sum'_n will be used to denote summation over all integer values of n from $-\infty$ to ∞ , excluding $n = 0$.

The ${}_1\psi_1$ function is defined to be

$${}_1\psi_1(a; b; q, x) = \sum_n \frac{(a; q)_n}{(b; q)_n} x^n.$$

Ramanujan's ${}_1\psi_1$ summation formula is

$$\sum_n \frac{(a; q)_n}{(b; q)_n} x^n = \frac{(ax, q/ax, q, b/a; q)_\infty}{(x, b/ax, b, q/a; q)_\infty}. \quad (1.1)$$

For a proof of this result and for additional information about the ${}_1\psi_1$ function, please see [1, equation (3.15)].

The Jordan-Kronecker function, which is introduced by Venkatachaliengar on p.37, is a special

case of the ${}_1\psi_1$ function, and is defined as follows.

Definition Let

$$f(a, t) = \sum_{n=-\infty}^{\infty} \frac{t^n}{1 - aq^{2n}}. \quad (1.2)$$

This series converges provided $|q^2| < |t| < 1$, and so long as $a \neq q^{2k}$, $k = 0, \pm 1, \pm 2, \dots$. Using the ${}_1\psi_1$ summation formula (1.1) we obtain

$$f(a, t) = \frac{(at, q^2/at, q^2, q^2; q^2)_{\infty}}{(t, q^2/t, a, q^2/a; q^2)_{\infty}}. \quad (1.3)$$

This extends the definition of f to all values of a and t except for $a, t = q^{2k}$, $k = 0, \pm 1, \pm 2, \dots$, where there are simple poles. The following results are immediate from (1.3):

$$f(a, t) = f(t, a), \quad (1.4)$$

$$f(a, t) = -f(1/a, 1/t), \quad (1.5)$$

$$f(a, t) = tf(aq^2, t) = af(a, tq^2). \quad (1.6)$$

The twelve Jacobian elliptic functions correspond to the twelve functions $f(A, Be^{i\theta})$, where $A = -1, q$ or $-q$, and $B = 1, -1, q$ or $-q$. The precise identifications will be given at the end of these notes. The Weierstrass σ and \wp functions are also related to the Jordan-Kronecker function f , and some of these connections will be given in sections 3 and 4.

At the beginning of chapter 3 of [2], Venkatachaliengar derives a fundamental multiplicative identity for the function f . He uses this identity to develop the theory of the Weierstrass and Jacobian elliptic functions. These notes describe how to obtain results such as the differential equations and addition formulas for the Jacobian elliptic functions, the connection between the Weierstrass \wp function and the Jacobian elliptic functions, and the differential equation for the \wp function, from Venkatachaliengar's fundamental multiplicative identity.

These notes deal with only a small part of the theory of elliptic functions. Topics such as hypergeometric functions, modular transformations and the problem of inversion are not even mentioned here. These topics, however, are taken up and developed in Venkatachaliengar's book.

2 The fundamental multiplicative identity and the Weierstrass \wp function

Venkatachaliengar's development of elliptic functions is based on the following result (see [2, p. 37]).

Theorem (Fundamental multiplicative identity)

$$f(a, t)f(b, t) = t \frac{\partial}{\partial t} f(ab, t) + f(ab, t)(\rho_1(a) + \rho_1(b)), \quad (2.1)$$

where the function ρ_1 is defined by

$$\rho_1(z) = \frac{1}{2} + \sum'_n \frac{z^n}{1 - q^{2n}}. \quad (2.2)$$

Remark

The series (2.2) defining ρ_1 converges in the annulus $|q^2| < |z| < 1$. Shortly we will obtain the

analytic continuation of ρ_1 , so the identity (2.1) will be valid for all values of a, b and t .

Proof

For $|q^2| < |a|, |b| < 1$, we have

$$\begin{aligned} f(a, t)f(b, t) &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{a^m b^n}{(1 - tq^{2m})(1 - tq^{2n})} \\ &= \sum_{m=-\infty}^{\infty} \frac{a^m b^m}{(1 - tq^{2m})^2} + \sum_{m \neq n} \frac{a^m b^n}{(1 - tq^{2m})(1 - tq^{2n})}. \end{aligned} \quad (2.3)$$

The first sum is

$$\begin{aligned} \sum_{m=-\infty}^{\infty} \frac{(ab)^m}{(1 - tq^{2m})^2} &= \sum_{m=-\infty}^{\infty} \frac{\partial}{\partial t} \frac{(ab/q^2)^m}{(1 - tq^{2m})} \\ &= \frac{\partial}{\partial t} f(ab/q^2, t) \\ &= \frac{\partial}{\partial t} [tf(ab, t)] \\ &= t \frac{\partial}{\partial t} f(ab, t) + f(ab, t) \end{aligned} \quad (2.4)$$

The penultimate step above follows from (1.6). The interchange of differentiation and summation is valid as all series converge absolutely and uniformly (in t) on compact sets which avoid the poles $t = q^{2k}$, $k = 0, \pm 1, \pm 2, \dots$, provided $|q^4| < |ab| < |q^2|$. By analytic continuation, equation (2.4) continues to remain valid for $|q^4| < |ab| < 1$.

Using partial fractions, the second sum on the right hand side of (2.3) becomes

$$\begin{aligned} &\sum_{m \neq n} \frac{a^m b^n}{(1 - tq^{2m})(1 - tq^{2n})} \\ &= \sum_{m=-\infty}^{\infty} \sum_k' \frac{a^m b^{m+k}}{(1 - tq^{2m})(1 - tq^{2m+2k})} \\ &= \sum_{m=-\infty}^{\infty} \sum_k' \frac{a^m b^{m+k}}{(1 - tq^{2m})(1 - q^{2k})} + \sum_{m=-\infty}^{\infty} \sum_k' \frac{a^m b^{m+k}}{(1 - tq^{2m+2k})(1 - q^{-2k})} \\ &= \sum_{m=-\infty}^{\infty} \frac{a^m b^m}{(1 - tq^{2m})} \sum_k' \frac{b^k}{(1 - q^{2k})} + \sum_k' \frac{a^{-k}}{(1 - q^{-2k})} \sum_{m=-\infty}^{\infty} \frac{a^{m+k} b^{m+k}}{(1 - tq^{2m+2k})} \\ &= f(ab, t) \left[\sum_k' \frac{a^k}{1 - q^{2k}} + \sum_k' \frac{b^k}{1 - q^{2k}} \right] \\ &= f(ab, t) [\rho_1(a) + \rho_1(b) - 1]. \end{aligned} \quad (2.5)$$

All of the series in the derivation of (2.5) converge at least for $|q| < |a|, |b| < 1$ and $t \neq q^{2k}$, $k = 0, \pm 1, \pm 2, \dots$, and so the series rearrangements above are valid. Now combine (2.3), (2.4) and (2.5). This gives (2.1) and proves the theorem.

The analytic continuation of ρ_1 can be obtained as follows.

$$\rho_1(z) = \frac{1}{2} + \sum_n' \frac{z^n}{1 - q^{2n}}$$

$$\begin{aligned}
&= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{z^n}{1-q^{2n}} + \sum_{n=1}^{\infty} \frac{z^{-n}}{1-q^{-2n}} \\
&= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{z^n(1-q^{2n}+q^{2n})}{1-q^{2n}} - \sum_{n=1}^{\infty} \frac{z^{-n}q^{2n}}{1-q^{2n}} \\
&= \frac{1}{2} + \frac{z}{1-z} + \sum_{n=1}^{\infty} \frac{z^n q^{2n}}{1-q^{2n}} - \sum_{n=1}^{\infty} \frac{z^{-n} q^{2n}}{1-q^{2n}} \\
&= \frac{1+z}{2(1-z)} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (z^n q^{2mn} - z^{-n} q^{2mn}) \\
&= \frac{1+z}{2(1-z)} + \sum_{m=1}^{\infty} \left(\frac{z q^{2m}}{1-z q^{2m}} - \frac{z^{-1} q^{2m}}{1-z^{-1} q^{2m}} \right). \tag{2.6}
\end{aligned}$$

This last series converges for all values of z except $z = q^{2k}$, $k = 0, \pm 1, \pm 2, \dots$, where there are poles of order 1. Thus (2.6) gives the analytic continuation of the function ρ_1 , and so now the fundamental multiplicative identity (2.1) is valid for all values of a , b and t .

The function ρ_1 is related to the Weierstrass \wp function in the following way. The Weierstrass \wp function with periods 2π and $2\pi\tau$ is defined by

$$\wp(\theta) = \frac{1}{\theta^2} + \sum'_{m,n} \left[\frac{1}{(\theta - 2\pi n - 2\pi\tau m)^2} - \frac{1}{(2\pi n + 2\pi\tau m)^2} \right]. \tag{2.7}$$

The symbol $\sum'_{m,n}$ denotes a double sum over all integer values of m and n from $-\infty$ to ∞ , excluding $(m, n) = (0, 0)$. Using the results

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\theta - 2\pi n)^2} = \frac{1}{4 \sin^2 \frac{\theta}{2}}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

we have that

$$\begin{aligned}
\wp(\theta) &= \frac{1}{\theta^2} + \sum'_n \left[\frac{1}{(\theta - 2\pi n)^2} - \frac{1}{(2\pi n)^2} \right] \\
&\quad + \sum'_m \sum_{n=-\infty}^{\infty} \left[\frac{1}{(\theta - 2\pi n - 2\pi\tau m)^2} - \frac{1}{(2\pi n + 2\pi\tau m)^2} \right] \\
&= \sum_n \frac{1}{(\theta - 2\pi n)^2} - \frac{2}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&\quad + \sum'_m \sum_{n=-\infty}^{\infty} \left[\frac{1}{(\theta - 2\pi n - 2\pi\tau m)^2} - \frac{1}{(2\pi n + 2\pi\tau m)^2} \right] \\
&= \frac{1}{4 \sin^2 \frac{\theta}{2}} - \frac{1}{12} + \sum'_m \left[\frac{1}{4 \sin^2(\frac{\theta}{2} - \pi\tau m)} - \frac{1}{4 \sin^2 \pi\tau m} \right] \\
&= -\frac{1}{12} - \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{\sin^2 \pi\tau m} + \frac{1}{4} \sum_{m=-\infty}^{\infty} \frac{1}{\sin^2(\frac{\theta}{2} + \pi\tau m)}.
\end{aligned}$$

Recall that $q = e^{i\pi\tau}$. Then

$$\begin{aligned}\wp(\theta) &= -\frac{1}{12} + 2 \sum_{m=1}^{\infty} \frac{1}{(q^m - q^{-m})^2} - \sum_{m=-\infty}^{\infty} \frac{1}{(e^{i\theta/2}q^m - e^{-i\theta/2}q^{-m})^2} \\ &= -\frac{1}{12} + 2 \sum_{m=1}^{\infty} \frac{q^{2m}}{(1 - q^{2m})^2} - \sum_{m=-\infty}^{\infty} \frac{e^{i\theta}q^{2m}}{(1 - e^{i\theta}q^{2m})^2}.\end{aligned}\quad (2.8)$$

Continuing, we have

$$\begin{aligned}\wp(\theta) &= -\frac{1}{12} + 2 \sum_{m=1}^{\infty} \frac{q^{2m}}{(1 - q^{2m})^2} \\ &\quad - \frac{e^{i\theta}}{(1 - e^{i\theta})^2} - \sum_{m=1}^{\infty} \left[\frac{e^{i\theta}q^{2m}}{(1 - e^{i\theta}q^{2m})^2} + \frac{e^{-i\theta}q^{2m}}{(1 - e^{-i\theta}q^{2m})^2} \right] \\ &= -\frac{1}{12} + 2 \sum_{m=1}^{\infty} \frac{q^{2m}}{(1 - q^{2m})^2} + i \frac{d}{d\theta} \rho_1(e^{i\theta}).\end{aligned}$$

Formula (2.6) was used to obtain the last line. Thus if we let

$$P = 1 - 24 \sum_{m=1}^{\infty} \frac{q^{2m}}{(1 - q^{2m})^2}, \quad (2.9)$$

then we have

$$\wp(\theta) = i \frac{d}{d\theta} \rho_1(e^{i\theta}) - \frac{P}{12}. \quad (2.10)$$

3 Jacobian elliptic functions

We will now look at three special cases of the function $f(a, t)$, which we shall call f_1 , f_2 and f_3 . These functions will turn out to be the Jacobian elliptic functions cs , ns and ds , respectively, up to rescaling. The precise identifications will be given at the end of the notes.

A number of properties of f_1 , f_2 and f_3 (Fourier series, infinite product formulas, double periodicity, location of zeros and poles) will follow immediately from the definition of $f(a, t)$ and Ramanujan's ${}_1\psi_1$ summation formula. We will then use the fundamental multiplicative identity (2.1) to obtain some of the other properties of these functions, namely the connection with the \wp function, elliptic analogues of the formula $\sin^2 \theta + \cos^2 \theta = 1$, derivatives and addition formulas.

Definition Let

$$f_1(\theta) = \frac{1}{i} f(e^{i\pi}, e^{i\theta}), \quad (3.1)$$

$$f_2(\theta) = \frac{e^{i\theta/2}}{i} f(e^{i\pi\tau}, e^{i\theta}), \quad (3.2)$$

$$f_3(\theta) = \frac{e^{i\theta/2}}{i} f(e^{i\pi+i\pi\tau}, e^{i\theta}). \quad (3.3)$$

The factors $1/i$ and $e^{i\theta/2}/i$ are included so that f_1 , f_2 and f_3 will be real valued when θ is real. The Fourier expansions follow directly from (1.2), the definition of f . For example

$$f_1(\theta) = \frac{1}{i} \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{1 + q^{2n}}$$

$$\begin{aligned}
&= \frac{1}{i} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{e^{in\theta}}{1+q^{2n}} + \sum_{n=1}^{\infty} \frac{e^{-in\theta}}{1+q^{-2n}} \right] \\
&= \frac{1}{i} \left[\frac{1}{2} + \sum_{n=1}^{\infty} e^{in\theta} - \sum_{n=1}^{\infty} \frac{q^{2n} e^{in\theta}}{1+q^{2n}} + \sum_{n=1}^{\infty} \frac{q^{2n} e^{-in\theta}}{1+q^{2n}} \right] \\
&= \frac{1}{i} \left[\frac{1}{2} + \frac{e^{i\theta}}{1-e^{i\theta}} - \sum_{n=1}^{\infty} \frac{q^{2n}}{1+q^{2n}} (e^{in\theta} - e^{-in\theta}) \right] \\
&= \frac{1}{2} \cot \frac{\theta}{2} - 2 \sum_{n=1}^{\infty} \frac{q^{2n}}{1+q^{2n}} \sin n\theta.
\end{aligned} \tag{3.4}$$

Similarly,

$$\begin{aligned}
f_2(\theta) &= \frac{e^{i\theta/2}}{i} \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{1-q^{2n+1}} \\
&= \frac{1}{2} \csc \frac{\theta}{2} + 2 \sum_{n=0}^{\infty} \frac{q^{2n+1}}{1-q^{2n+1}} \sin(n + \frac{1}{2})\theta,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
f_3(\theta) &= \frac{e^{i\theta/2}}{i} \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{1+q^{2n+1}} \\
&= \frac{1}{2} \csc \frac{\theta}{2} - 2 \sum_{n=0}^{\infty} \frac{q^{2n+1}}{1+q^{2n+1}} \sin(n + \frac{1}{2})\theta.
\end{aligned} \tag{3.6}$$

Infinite product formulas follow from (1.3). We find that

$$f_1(\theta) = \frac{1}{i} \frac{(-e^{i\theta}, -q^2 e^{-i\theta}, q^2, q^2; q^2)_{\infty}}{(e^{i\theta}, q^2 e^{-i\theta}, -1, -q^2; q^2)_{\infty}} \tag{3.7}$$

$$= \frac{1}{2} \frac{(q^2; q^2)_{\infty}^2}{(-q^2; q^2)_{\infty}^2} \cot \frac{\theta}{2} \prod_{n=1}^{\infty} \frac{(1 + 2q^{2n} \cos \theta + q^{4n})}{(1 - 2q^{2n} \cos \theta + q^{4n})}, \tag{3.8}$$

$$f_2(\theta) = \frac{e^{i\theta/2}}{i} \frac{(qe^{i\theta}, qe^{-i\theta}, q^2, q^2; q^2)_{\infty}}{(e^{i\theta}, q^2 e^{-i\theta}, q, q; q^2)_{\infty}} \tag{3.9}$$

$$= \frac{1}{2} \frac{(q^2; q^2)_{\infty}^2}{(q; q^2)_{\infty}^2} \csc \frac{\theta}{2} \prod_{n=1}^{\infty} \frac{(1 - 2q^{2n-1} \cos \theta + q^{4n-2})}{(1 - 2q^{2n} \cos \theta + q^{4n})}, \tag{3.10}$$

$$f_3(\theta) = \frac{e^{i\theta/2}}{i} \frac{(-qe^{i\theta}, -qe^{-i\theta}, q^2, q^2; q^2)_{\infty}}{(e^{i\theta}, q^2 e^{-i\theta}, -q, -q; q^2)_{\infty}} \tag{3.11}$$

$$= \frac{1}{2} \frac{(q^2; q^2)_{\infty}^2}{(-q; q^2)_{\infty}^2} \csc \frac{\theta}{2} \prod_{n=1}^{\infty} \frac{(1 + 2q^{2n-1} \cos \theta + q^{4n-2})}{(1 - 2q^{2n} \cos \theta + q^{4n})}. \tag{3.12}$$

Either from the Fourier series (3.4) or from the infinite product formula (3.7), we see that $f_1(\theta + 2\pi) = f_1(\theta)$.

From the infinite product (3.7), we obtain

$$\begin{aligned}
\frac{f_1(\theta + 2\pi\tau)}{f_1(\theta)} &= \frac{(-q^2 e^{i\theta}, -e^{-i\theta}; q^2)_{\infty}}{(q^2 e^{i\theta}, e^{-i\theta}; q^2)_{\infty}} \div \frac{(-e^{i\theta}, -q^2 e^{-i\theta}; q^2)_{\infty}}{(e^{i\theta}, q^2 e^{-i\theta}; q^2)_{\infty}} \\
&= \frac{(1 + e^{-i\theta})}{(1 + e^{i\theta})} \frac{(1 - e^{i\theta})}{(1 - e^{-i\theta})} = -1,
\end{aligned}$$

and therefore $f_1(\theta + 2\pi\tau) = -f_1(\theta)$. Similar calculations can be done for f_2 and f_3 . The results are summarized below.

$$f_1(\theta + 2\pi m + 2\pi\tau n) = (-1)^n f_1(\theta), \tag{3.13}$$

$$f_2(\theta + 2\pi m + 2\pi\tau n) = (-1)^m f_2(\theta), \quad (3.14)$$

$$f_3(\theta + 2\pi m + 2\pi\tau n) = (-1)^{m+n} f_3(\theta). \quad (3.15)$$

Here m and n are integers. Thus f_1 is doubly periodic with periods 2π and $4\pi\tau$, f_2 is doubly periodic with periods 4π and $2\pi\tau$, while f_3 is doubly periodic with periods 4π and $2\pi + 2\pi\tau$.

From the infinite product expansion (3.7) we see that f_1 has zeros when $1 + q^{2n}e^{i\theta} = 0$, where n is any integer. Remembering that $q = e^{i\pi\tau}$, this implies that $f_1(\theta) = 0$ when $\theta = (2m+1)\pi + 2n\pi\tau$, for any integer values of m and n . The zeros of f_2 and f_3 are at $\theta = 2m\pi + (2n+1)\pi\tau$ and $\theta = (2m+1)\pi + (2n+1)\pi\tau$, respectively. The poles of f_1 , f_2 and f_3 all occur when $1 - q^{2n}e^{i\theta} = 0$, that is, when $\theta = 2m\pi + 2n\pi\tau$.

Before describing the connection of f_1 , f_2 and f_3 with the Weierstrass \wp function, we define the Weierstrass invariants e_1 , e_2 and e_3 .

Definition Let

$$e_1 = \wp(\pi), \quad (3.16)$$

$$e_2 = \wp(\pi\tau), \quad (3.17)$$

$$e_3 = \wp(\pi + \pi\tau). \quad (3.18)$$

Explicit formulas for e_1 , e_2 and e_3 follow at once from equation (2.8). Specifically,

$$e_1 = \frac{1}{6} + 2 \sum_{m=1}^{\infty} \frac{q^{2m}}{(1 - q^{2m})^2} + 2 \sum_{m=1}^{\infty} \frac{q^{2m}}{(1 + q^{2m})^2}, \quad (3.19)$$

$$e_2 = -\frac{1}{12} + 2 \sum_{m=1}^{\infty} \frac{q^{2m}}{(1 - q^{2m})^2} - 2 \sum_{m=1}^{\infty} \frac{q^{2m-1}}{(1 - q^{2m-1})^2}, \quad (3.20)$$

$$e_3 = -\frac{1}{12} + 2 \sum_{m=1}^{\infty} \frac{q^{2m}}{(1 - q^{2m})^2} + 2 \sum_{m=1}^{\infty} \frac{q^{2m-1}}{(1 + q^{2m-1})^2}. \quad (3.21)$$

Relation of f_1 , f_2 and f_3 to the Weierstrass \wp function

Let $b \rightarrow 1/a$ in the fundamental identity (2.1):

$$\lim_{b \rightarrow 1/a} f(a, t) f(b, t) = \lim_{b \rightarrow 1/a} t \frac{\partial}{\partial t} f(ab, t) + \lim_{b \rightarrow 1/a} f(ab, t) (\rho_1(a) + \rho_1(b)). \quad (3.22)$$

The left hand side is just $f(a, t) f(1/a, t)$.

The first limit on the right hand side is

$$\begin{aligned} \lim_{b \rightarrow 1/a} t \frac{\partial}{\partial t} \sum_{n=-\infty}^{\infty} \frac{t^n}{1 - abq^{2n}} &= \lim_{b \rightarrow 1/a} \sum_n' \frac{nt^n}{1 - abq^{2n}} \\ &= \sum_n' \frac{nt^n}{1 - q^{2n}} = t \frac{d}{dt} \rho_1(t). \end{aligned}$$

From equation (2.6) it follows that $\rho_1(b) = -\rho_1(1/b)$. Using this and the infinite product formula (1.3) for the function f , the remaining limit on the right hand side of equation (3.22) becomes

$$\lim_{b \rightarrow 1/a} f(ab, t) (\rho_1(a) + \rho_1(b))$$

$$\begin{aligned}
&= \lim_{b \rightarrow 1/a} (1 - ab) f(ab, t) \lim_{b \rightarrow 1/a} \frac{\rho_1(a) + \rho_1(b)}{1 - ab} \\
&= \lim_{b \rightarrow 1/a} (1 - ab) \frac{(abt, q^2/abt, q^2, q^2; q^2)_\infty}{(t, q^2/t, ab, q^2/ab; q^2)_\infty} \lim_{b \rightarrow 1/a} \frac{\rho_1(a) - \rho_1(1/b)}{a - 1/b} \left(-\frac{1}{b}\right) \\
&= (1)\rho'_1(a)(-a).
\end{aligned}$$

Thus

$$f(a, t)f(1/a, t) = t \frac{d}{dt} \rho_1(t) - a \frac{d}{da} \rho_1(a).$$

On letting $a = e^{i\alpha}$, $t = e^{i\theta}$ and using equation (2.10), this becomes

$$f(e^{i\alpha}, e^{i\theta})f(e^{-i\alpha}, e^{i\theta}) = \wp(\alpha) - \wp(\theta). \quad (3.23)$$

Remark

This formula can also be obtained by combining the two terms on the right hand side of (3.23) into a single series using (2.8), and then applying the ${}_6\psi_6$ summation formula.

Letting $\alpha = \pi$, $\alpha = \pi\tau$ and $\alpha = \pi + \pi\tau$ in (3.23), respectively, and simplifying, gives

$$f_1^2(\theta) = \wp(\theta) - e_1, \quad (3.24)$$

$$f_2^2(\theta) = \wp(\theta) - e_2, \quad (3.25)$$

$$f_3^2(\theta) = \wp(\theta) - e_3. \quad (3.26)$$

Successively letting $\theta = \pi\tau$ in (3.24), $\theta = \pi + \pi\tau$ in (3.25) and $\theta = \pi$ in (3.26), and using the infinite products for f_1 , f_2 and f_3 , gives

$$e_1 - e_2 = \frac{1}{4} \frac{(-q; q^2)_\infty^4 (q^2; q^2)_\infty^4}{(q; q^2)_\infty^4 (-q^2; q^2)_\infty^4} \quad (3.27)$$

$$e_3 - e_2 = 4q \frac{(-q^2; q^2)_\infty^4 (q^2; q^2)_\infty^4}{(-q; q^2)_\infty^4 (q; q^2)_\infty^4} \quad (3.28)$$

$$e_1 - e_3 = \frac{1}{4} \frac{(q; q^2)_\infty^4 (q^2; q^2)_\infty^4}{(-q^2; q^2)_\infty^4 (-q; q^2)_\infty^4} \quad (3.29)$$

Note that since $\text{Im } \tau > 0$ this implies that $e_1 \neq e_2 \neq e_3 \neq e_1$. Further, if τ is purely imaginary, then q is real, and so in this case we also have $e_1 > e_3 > e_2$.

If we let

$$x = \frac{e_3 - e_2}{e_1 - e_2} = 16q \frac{(-q^2; q^2)_\infty^8}{(-q; q^2)_\infty^8}, \quad (3.30)$$

$$x' = \frac{e_1 - e_3}{e_1 - e_2} = \frac{(q; q^2)_\infty^8}{(-q; q^2)_\infty^8}, \quad (3.31)$$

then clearly $x + x' = 1$, and hence we obtain Jacobi's formula

$$(q; q^2)_\infty^8 + 16q(-q^2; q^2)_\infty^8 = (-q; q^2)_\infty^8.$$

If the equations (3.24), (3.25) and (3.26) are combined two at a time to eliminate the $\wp(\theta)$ term, we obtain

$$f_2^2(\theta) - f_1^2(\theta) = e_1 - e_2, \quad (3.32)$$

$$f_2^2(\theta) - f_3^2(\theta) = e_3 - e_2, \quad (3.33)$$

$$f_3^2(\theta) - f_1^2(\theta) = e_1 - e_3. \quad (3.34)$$

These are the elliptic function analogues of the trigonometric identity $\sin^2 \theta + \cos^2 \theta = 1$. In fact, from (3.4)–(3.6) and (3.19)–(3.21), we have

$$\lim_{q \rightarrow 0} f_1(\theta) = \frac{1}{2} \cot \frac{\theta}{2}, \quad \lim_{q \rightarrow 0} f_2(\theta) = \lim_{q \rightarrow 0} f_3(\theta) = \frac{1}{2} \csc \frac{\theta}{2},$$

$$\lim_{q \rightarrow 0} e_1 = 1/6, \quad \lim_{q \rightarrow 0} e_2 = \lim_{q \rightarrow 0} e_3 = -1/12.$$

Therefore when $q = 0$, (3.32) and (3.34) reduce to

$$\frac{1}{4} \csc^2 \frac{\theta}{2} - \frac{1}{4} \cot^2 \frac{\theta}{2} = \frac{1}{4},$$

while (3.33) reduces to a tautology.

Derivatives

In the fundamental multiplicative identity (2.1), let $t = e^{i\theta}$ to get

$$f(a, e^{i\theta})f(b, e^{i\theta}) = \frac{1}{i} \frac{\partial}{\partial \theta} f(ab, e^{i\theta}) + f(ab, e^{i\theta})(\rho_1(a) + \rho_1(b)). \quad (3.35)$$

Now let $a = e^{i\pi}$ and $b = e^{i\pi\tau}$. From (2.6) we have $\rho_1(e^{i\pi}) = 0$, $\rho_1(e^{i\pi\tau}) = \frac{1}{2}$, hence

$$f(-1, e^{i\theta})f(q, e^{i\theta}) = \frac{1}{i} \frac{\partial}{\partial \theta} f(-q, e^{i\theta}) + \frac{1}{2} f(-q, e^{i\theta}). \quad (3.36)$$

The left hand side of this is

$$f(-1, e^{i\theta})f(q, e^{i\theta}) = if_1(\theta)ie^{-i\theta/2}f_2(\theta) = -e^{-i\theta/2}f_1(\theta)f_2(\theta).$$

The right hand side of (3.36) is

$$\begin{aligned} & \frac{1}{i} \frac{\partial}{\partial \theta} \left(ie^{-i\theta/2} f_3(\theta) \right) + \frac{i}{2} e^{-i\theta/2} f_3(\theta) \\ &= e^{-i\theta/2} f_3'(\theta) - \frac{i}{2} e^{-i\theta/2} f_3(\theta) + \frac{i}{2} e^{-i\theta/2} f_3(\theta) \\ &= e^{-i\theta/2} f_3'(\theta). \end{aligned}$$

Combining gives

$$f_3'(\theta) = -f_1(\theta)f_2(\theta). \quad (3.37)$$

Similarly, letting $a = e^{i\pi\tau}$, $b = e^{i\pi+i\pi\tau}$ and $a = e^{i\pi+i\pi\tau}$, $b = e^{i\pi}$ in (3.35) leads, respectively, to

$$f_1'(\theta) = -f_2(\theta)f_3(\theta), \quad (3.38)$$

$$f_2'(\theta) = -f_3(\theta)f_1(\theta). \quad (3.39)$$

Venkatachaliengar shows how to obtain the differential equation for the \wp function from the fundamental multiplicative identity. We will instead obtain it by putting together the previous results. From (3.24) we have that

$$\wp(\theta) = e_1 + f_1^2(\theta).$$

Differentiate both sides and use (3.38) to simplify the result.

$$\wp'(\theta) = 2f_1(\theta)f_1'(\theta) = -2f_1(\theta)f_2(\theta)f_3(\theta).$$

Therefore, by (3.24), (3.25) and (3.26), we have

$$(\wp'(\theta))^2 = 4f_1^2(\theta)f_2^2(\theta)f_3^2(\theta) = 4(\wp(\theta) - e_1)(\wp(\theta) - e_2)(\wp(\theta) - e_3). \quad (3.40)$$

Addition formulas

The fundamental multiplicative identity (2.1) can be written in the form

$$f(e^{i\alpha}, e^{i\theta})f(e^{i\beta}, e^{i\theta}) = \frac{1}{i}f(e^{i(\alpha+\beta)}, e^{i\theta}) + f(e^{i(\alpha+\beta)}, e^{i\theta})(\rho_1(e^{i\alpha}) + \rho_1(e^{i\beta})).$$

Apply $\partial/\partial\alpha - \partial/\partial\beta$ to both sides. The result is

$$\begin{aligned} & \frac{\partial}{\partial\alpha}f(e^{i\alpha}, e^{i\theta})f(e^{i\beta}, e^{i\theta}) - \frac{\partial}{\partial\beta}f(e^{i\alpha}, e^{i\theta})f(e^{i\beta}, e^{i\theta}) \\ &= f(e^{i(\alpha+\beta)}, e^{i\theta}) \left(\frac{d}{d\alpha}\rho_1(e^{i\alpha}) - \frac{d}{d\beta}\rho_1(e^{i\beta}) \right). \end{aligned}$$

Rearranging this and using (2.10) gives

$$f(e^{i(\alpha+\beta)}, e^{i\theta}) = \frac{i \left[\frac{\partial}{\partial\alpha}f(e^{i\alpha}, e^{i\theta})f(e^{i\beta}, e^{i\theta}) - \frac{\partial}{\partial\beta}f(e^{i\alpha}, e^{i\theta})f(e^{i\beta}, e^{i\theta}) \right]}{\wp(\alpha) - \wp(\beta)}. \quad (3.41)$$

Let $\theta = \pi$ in this to get

$$if_1(\alpha + \beta) = \frac{i [if_1'(\alpha)if_1(\beta) - if_1(\alpha)if_1'(\beta)]}{\wp(\alpha) - \wp(\beta)}.$$

Simplify this using (3.24) and (3.38). The result is

$$f_1(\alpha + \beta) = \frac{f_1(\alpha)f_2(\beta)f_3(\beta) - f_1(\beta)f_2(\alpha)f_3(\alpha)}{f_1^2(\beta) - f_1^2(\alpha)}. \quad (3.42)$$

Similarly, letting $\theta = \pi\tau$ and $\theta = \pi + \pi\tau$ in (3.41) leads to

$$f_2(\alpha + \beta) = \frac{f_2(\alpha)f_3(\beta)f_1(\beta) - f_2(\beta)f_3(\alpha)f_1(\alpha)}{f_2^2(\beta) - f_2^2(\alpha)}, \quad (3.43)$$

$$f_3(\alpha + \beta) = \frac{f_3(\alpha)f_1(\beta)f_2(\beta) - f_3(\beta)f_1(\alpha)f_2(\alpha)}{f_3^2(\beta) - f_3^2(\alpha)}. \quad (3.44)$$

The fundamental multiplicative identity (2.1) can also be used to derive addition formulas for the Weierstrass \wp function. Venkatachaliengar's derivation of the symmetric form of the addition formula for the \wp function goes as follows.

Let $t = e^v$ and write the Jordan–Kronecker function as

$$\begin{aligned} f(a, t) &= f(a, e^v) = \sum_{n=-\infty}^{\infty} \frac{e^{nv}}{1 - aq^{2n}} \\ &= \frac{1}{1-a} + \sum_{n=1}^{\infty} \frac{e^{nv}(1 - aq^{2n} + aq^{2n})}{1 - aq^{2n}} + \sum_{n=1}^{\infty} \frac{e^{-nv}}{1 - aq^{-2n}} \\ &= \frac{1}{1-a} - \frac{1}{e^v - 1} - 1 + \sum_{n=1}^{\infty} \left(\frac{e^{nv}aq^{2n}}{1 - aq^{2n}} - \frac{e^{-nv}a^{-1}q^{2n}}{1 - a^{-1}q^{2n}} \right). \end{aligned} \quad (3.45)$$

The series (3.45) converges for $|\operatorname{Re} v| < \operatorname{Im} 2\pi\tau$. Hence in the annulus $0 < |v| < \min\{2\pi, \operatorname{Im} 2\pi\tau\}$, the function $f(a, e^v)$ can be expanded further as a Laurent series in powers of v . Since

$$\frac{v}{e^v - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} v^k,$$

where B_k are the Bernoulli numbers, we have

$$\begin{aligned}
& f(a, e^v) \\
&= \frac{1}{1-a} - 1 - \frac{1}{v} \sum_{k=0}^{\infty} \frac{B_k}{k!} v^k + \sum_{k=0}^{\infty} \frac{v^k}{k!} \sum_{n=1}^{\infty} \left(\frac{n^k a q^{2n}}{1-aq^{2n}} - \frac{(-1)^k n^k a^{-1} q^{2n}}{1-a^{-1} q^{2n}} \right) \\
&= -\frac{1}{v} + \left(-\frac{1}{2} + \frac{1}{1-a} + \sum_{n=1}^{\infty} \frac{a q^{2n}}{1-aq^{2n}} - \frac{a^{-1} q^{2n}}{1-a^{-1} q^{2n}} \right) \\
&\quad + \sum_{k=1}^{\infty} \frac{v^k}{k!} \left(-\frac{B_{k+1}}{k+1} + \sum_{n=1}^{\infty} \frac{n^k a q^{2n}}{1-aq^{2n}} - \frac{(-1)^k n^k a^{-1} q^{2n}}{1-a^{-1} q^{2n}} \right). \tag{3.46}
\end{aligned}$$

The term independent of v in this expansion is

$$-\frac{1}{2} + \frac{1}{1-a} + \sum_{n=1}^{\infty} \frac{a q^{2n}}{1-aq^{2n}} - \frac{a^{-1} q^{2n}}{1-a^{-1} q^{2n}}$$

which is precisely $\rho_1(a)$, the same function as in equation (2.6). The reason why ρ_1 occurs both here and in the fundamental multiplicative identity (2.1) will become clear below. For $v \geq 2$, let us define

$$\rho_k(a) = -\frac{B_k}{k} + \sum_{n=1}^{\infty} n^{k-1} \left[\frac{a q^{2n}}{1-aq^{2n}} - \frac{(-1)^{k-1} a^{-1} q^{2n}}{1-a^{-1} q^{2n}} \right]. \tag{3.47}$$

Then equation (3.46) becomes

$$f(a, e^v) = -\frac{1}{v} + \sum_{k=0}^{\infty} \frac{\rho_{k+1}(a) v^k}{k!}. \tag{3.48}$$

With $t = e^v$, the fundamental multiplicative identity can be written as

$$f(a, e^v) f(b, e^v) = \frac{\partial}{\partial v} f(ab, e^v) + f(ab, e^v) (\rho_1(a) + \rho_1(b)).$$

Substitute the expansion (3.48) into this to obtain

$$\begin{aligned}
& \left[-\frac{1}{v} + \sum_{k=0}^{\infty} \frac{\rho_{k+1}(a) v^k}{k!} \right] \left[-\frac{1}{v} + \sum_{k=0}^{\infty} \frac{\rho_{k+1}(b) v^k}{k!} \right] \\
&= \frac{1}{v^2} + \sum_{k=1}^{\infty} \frac{\rho_{k+1}(ab) v^{k-1}}{(k-1)!} + \left[-\frac{1}{v} + \sum_{k=0}^{\infty} \frac{\rho_{k+1}(ab) v^k}{k!} \right] (\rho_1(a) + \rho_1(b)). \tag{3.49}
\end{aligned}$$

Let us compare coefficients of v^k on both sides. Clearly the coefficients of v^{-2} are equal. The coefficients of v^{-1} are both equal to $-(\rho_1(a) + \rho_1(b))$. Thus if ρ_1 is defined to be the term independent of v in the expansion (3.48), then this explains why the term $\rho_1(a) + \rho_1(b)$ occurs on the right hand side of the fundamental multiplicative identity (2.1). Equating coefficients of v^0 in (3.49) gives

$$-\rho_2(a) - \rho_2(b) + \rho_1(a)\rho_1(b) = \rho_2(ab) + \rho_1(ab)(\rho_1(a) + \rho_1(b)). \tag{3.50}$$

From (2.6) and (3.47) we have $\rho_1(1/c) = -\rho_1(c)$ and $\rho_2(1/c) = \rho_2(c)$, respectively. Let $c = 1/ab$. Then (3.50) becomes

$$\rho_1(a)\rho_1(b) + \rho_1(b)\rho_1(c) + \rho_1(c)\rho_1(a) = \rho_2(a) + \rho_2(b) + \rho_2(c). \tag{3.51}$$

Venkatachaliengar gives a direct proof of this identity at the beginning of his book, and uses it to derive a number of results about the Weierstrass \wp function. Applying $a\partial/\partial a - b\partial/\partial b$ to both sides gives

$$a\rho_1'(a)(\rho_1(b) + \rho_1(c)) - b\rho_1'(b)(\rho_1(a) + \rho_1(c)) = a\rho_2'(a) - b\rho_2'(b).$$

Now apply $ab\partial^2/\partial a\partial b$ to this to obtain

$$\begin{aligned} & a^2b\rho_1''(a)\rho_1'(b) - ab^2\rho_1'(a)\rho_1''(b) \\ & + b^2c\rho_1''(b)\rho_1'(c) - bc^2\rho_1'(b)\rho_1''(c) \\ & + c^2a\rho_1''(c)\rho_1'(a) - ca^2\rho_1'(c)\rho_1''(a) = 0. \end{aligned}$$

This can also be written in the form

$$\det \begin{bmatrix} 1 & 1 & 1 \\ a\rho_1'(a) & b\rho_1'(b) & c\rho_1'(c) \\ a^2\rho_1''(a) & b^2\rho_1''(b) & c^2\rho_1''(c) \end{bmatrix} = 0.$$

Now let $a = e^{i\alpha}$, $b = e^{i\beta}$ and $c = e^{i\gamma}$, so that $\alpha + \beta + \gamma = 0$. Using (2.10) and elementary properties of determinants, this equivalent to

$$\det \begin{bmatrix} 1 & 1 & 1 \\ \wp(\alpha) & \wp(\beta) & \wp(\gamma) \\ \wp'(\alpha) & \wp'(\beta) & \wp'(\gamma) \end{bmatrix} = 0,$$

provided $\alpha + \beta + \gamma = 0$. This is the symmetric form of the addition formula for the Weierstrass \wp function. Venkatachaliengar also uses equation (3.51) to derive the addition formula in the form

$$\wp(\alpha + \beta) = \frac{1}{4} \left(\frac{\wp'(\alpha) - \wp'(\beta)}{\wp(\alpha) - \wp(\beta)} \right)^2 - \wp(\alpha) - \wp(\beta).$$

Please see [2, p. 9] for the details. This formula can also be obtained by taking the logarithm of (3.23) and then applying $\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}$ twice to both sides.

4 Identification with the notation and formulas in Whittaker and Watson

1. The modulus and complementary modulus.

The quantities x and x' defined in equations (3.30) and (3.31) correspond to the squares of the modulus k and complementary modulus k' , respectively. We have

$$\begin{aligned} x &= k^2 = 16q \frac{(-q^2; q^2)_\infty^8}{(-q; q^2)_\infty^8}, \\ x' &= k'^2 = \frac{(q; q^2)_\infty^8}{(-q; q^2)_\infty^8}. \end{aligned}$$

See [2, p. 86, eqn. (5.47)] and [3, p. 479 or p. 488, ex. 9,10].

2. Ramanujan's z and the complete elliptic integral K .

Although it was not introduced in these notes, Venkatachaliengar makes extensive use of the quantity z that was introduced by Ramanujan. We mention z now, to aid with the identification of f_1 , f_2 and f_3 with Jacobian elliptic functions.

$$z = \left(\sum_{n=-\infty}^{\infty} q^{n^2} \right)^2 = (-q; q^2)_\infty^4 (q^2; q^2)_\infty^2,$$

$$K = \frac{\pi}{2}z.$$

See [2, p. 86, eqn. (5.49)] and [3, p. 479].

3. f_1, f_2, f_3 and **Jacobian elliptic functions.**

On comparing the Fourier series (3.4)–(3.6) with those in [3, p. 511–512], we find

$$\begin{aligned} f_1(\theta) &= (K/\pi) \operatorname{cs}(K\theta/\pi, k), & \operatorname{cs}(u, k) &= (2/z)f_1(2u/z), \\ f_2(\theta) &= (K/\pi) \operatorname{ns}(K\theta/\pi, k), & \operatorname{ns}(u, k) &= (2/z)f_2(2u/z), \\ f_3(\theta) &= (K/\pi) \operatorname{ds}(K\theta/\pi, k), & \operatorname{ds}(u, k) &= (2/z)f_3(2u/z). \end{aligned}$$

4. **The functions $f(a, t), \rho_1(z)$ and the Weierstrass σ, ζ and \wp functions.**

Earlier, we showed that the Weierstrass \wp function with periods 2π and $2\pi\tau$ is related to the function ρ_1 by equation (2.10). Here is the formula again.

$$\wp(\theta) = i \frac{d}{d\theta} \rho_1(e^{i\theta}) - \frac{P}{12}.$$

The corresponding Weierstrassian ζ function is related to the function ρ_1 as follows.

$$\zeta(\theta) = -i\rho_1(e^{i\theta}) + \frac{P\theta}{12}.$$

See [3, pp. 445–447]. Recall that P is given by (2.9). In particular, this together with equation (2.6) gives

$$\begin{aligned} \eta_1 &= \zeta(\pi) = \frac{P\pi}{12}, \\ \eta_2 &= \zeta(\pi\tau) = -\frac{i}{2} + \frac{P\pi\tau}{12}, \end{aligned}$$

from which Legendre's identity (see [3, p. 446, sect. 20.411]) follows trivially:

$$\eta_1\pi\tau - \eta_2\pi = \frac{1}{2}\pi i.$$

The corresponding Weierstrass σ function is as follows.

$$\sigma(\theta) = ie^{-i\theta/2} e^{P\theta^2/24} \frac{(e^{i\theta}, q^2 e^{-i\theta}; q^2)_\infty}{(q^2; q^2)_\infty^2}.$$

See [3, pp. 447–448]. The Jordan–Kronecker function $f(a, t)$, (equation 1.2), is related to the Weierstrass σ function by

$$f(e^{i\alpha}, e^{i\theta}) = ie^{-P\alpha\theta/12} \frac{\sigma(\alpha + \theta)}{\sigma(\alpha)\sigma(\theta)}.$$

Location of the main formulas in Venkatachaliengar's book

1. Ramanujan's ${}_1\psi_1$ summation formula (1.1) is proved in [2, pp. 24–30].
2. The Jordan–Kronecker function $f(a, t)$, (equation 1.2), is defined in [2, p. 37]. The infinite product for $f(a, t)$, (equation 1.3), is [2, p. 40, eq. 3.32]. See also [3, p. 460, ex. 34], which is basically equation (1.3) in disguise. The fundamental multiplicative identity (2.1) is proved in [2, p. 41].
3. The function ρ_1 , (equation 2.2), is defined in [2, p. 5] and some of its properties, including the analytic continuation, are given there.

4. The connection between the Weierstrass \wp function and the function ρ_1 given by equation (2.10) is given in [2, p. 8, eq. 1.21]. Essentially the same formula is in [3, p. 460, ex. 35].
5. The functions f_1 , f_2 and f_3 , (equations 3.1–3.3), are defined in [2, p. 111]. The Fourier series for essentially the same functions are given in [3, pp. 511–512].
6. The Weierstrass invariants e_1 , e_2 and e_3 , (equations (3.16–3.18)), are defined in [2, p. 59].
7. Equation (3.23) is derived in [2, p. 112, eqn. 6.50]. It is equivalent to example 1 on p. 451 of [3].
8. Formulas (3.24)–(3.26) are in [2, p. 112]. Compare these with [3, p. 451 example 4] and [3, p. 505, section 22.351].
9. The infinite products (3.27)–(3.29) are given in [2, p. 66].
10. Formulas (3.30) and (3.31) for x and x' are in [2, p. 86, eqn. 5.47]. Analogous formulas for k and k' are in [3, p. 479 and p. 488, ex. 9, 10].
11. The derivatives of f_1 , f_2 and f_3 are computed in [2, p. 111].
12. The differential equation for the Weierstrass \wp function is in [2, p. 13, eqn. 1.49].
13. The addition formulas for f_1 , f_2 and f_3 are given in [2, pp. 112–113].
14. Formula (3.48) is given in [2, p. 43].
15. Formula (3.51) as derived here in the notes is given in [2, p. 43]. Also see [2, pp. 3–4].
16. Addition formulas for the \wp function are derived in [2, pp. 8–9].

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